Gromov hyperbolicity, geodesic defect, and apparent pairs in Vietoris-Rips filtrations

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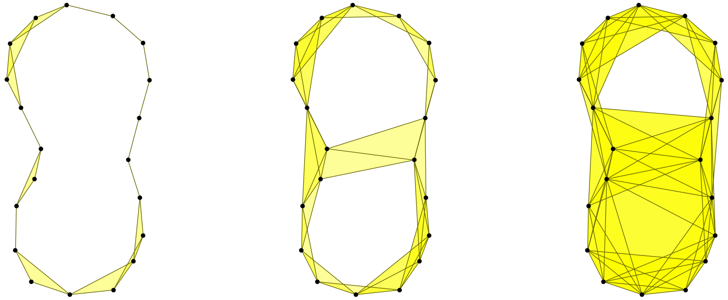
joint work with Ulrich Bauer

The Vietoris–Rips complex

Definition. Let X be a metric space. The Vietoris-Rips complex at scale r is the simplicial complex

$$\mathrm{Rips}_r(X) = \{S \subseteq X \text{ finite} \mid S \neq \varnothing,\ \operatorname{diam} S \leq r\}.$$



The Vietoris–Rips complex

Applications

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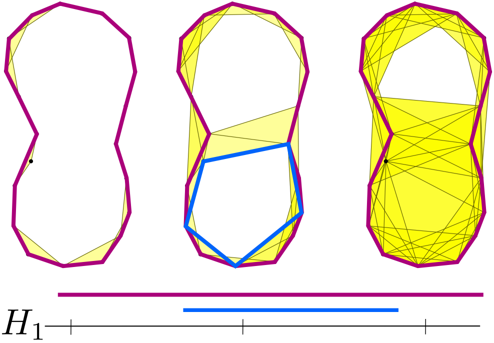
- In the limit $r \to \infty$: Used by Eliyahu Rips and Mikhael Gromov to study hyperbolic groups (1987).
- For all $r > 0$: Used in topological data analysis (nowadays).



Application of Vietoris–Rips persistent homology

COVID-19 genetic evolution data

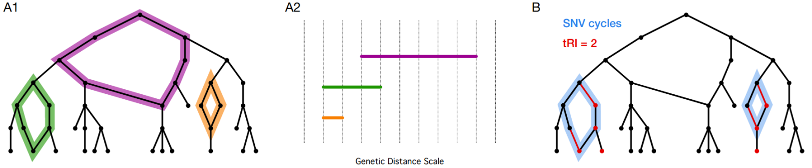


Figure 2. Topological data analysis quantifies convergent evolution. (A) Persistent homology detects reticulate events in viral evolution by means of a persistence barcode. Each bar in the barcode (A2) corresponds to a topological cycle in the reticulate phylogeny (A1). Bars at small genetic distance scales are expected to correspond mainly to homoplasies, while recombination events typically produce topological features at larger scales. (B) SNV cycles are topological cycles in the reticulate phylogeny for which adjacent sequences differ

M. Bleher, L. Hahn, J. A. Patino-Galindo, M. Carriere, U. Bauer, R. Rabadan, and A. Ott

Topological data analysis identifies emerging adaptive mutations in SARS-CoV-2

Preprint, arXiv:2106.07292, 2021

Application of Vietoris–Rips persistent homology

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covid data (≈ 15000 points)	Ripser's runtime
ordered chronologically	1 day

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Ripser: efficient computation of Vietoris–Rips persistence barcodes
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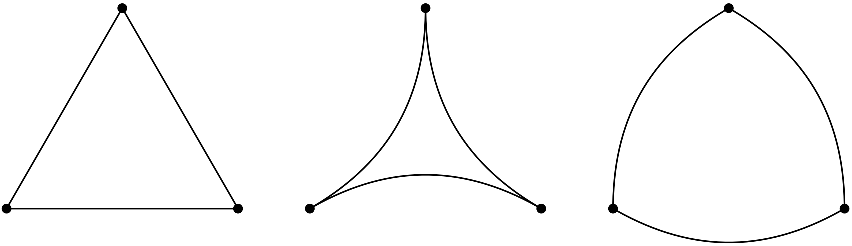
Rips contractibility lemma

Gromov-hyperbolicity



euclidean triangle hyperbolic triangle spherical triangle

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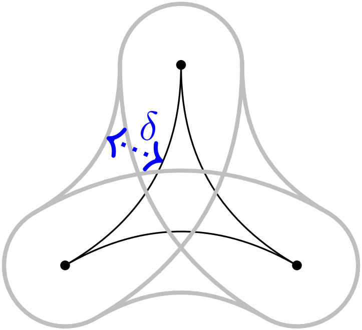

hyperbolic triangle

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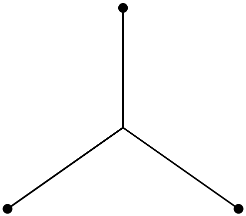

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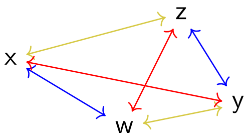
Gromov-hyperbolicity

Definition (four-point condition). A metric space X is (Gromov) δ-*hyperbolic* if for all four points $w, x, y, z \in X$

$$d(x,w) + d(y,z) \leq \max\{d(x,y) + d(z,w), d(x,z) + d(y,w)\} + 2\delta$$



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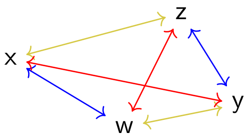

Example. finite metric spaces, trees are 0-hyperbolic, hyperbolic plane, ...

Rips contractibility lemma

Theorem (Rips, Gromov 1987). Let X be a δ-hyperbolic geodesic metric space. Then $\mathrm{Rips}_t(X)$ is contractible for all $t \geq 4\delta$.

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Any hyperbolic group G

- is finitely generated and finitely presented.
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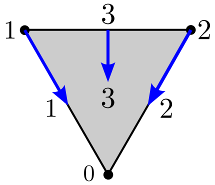
We address two questions:

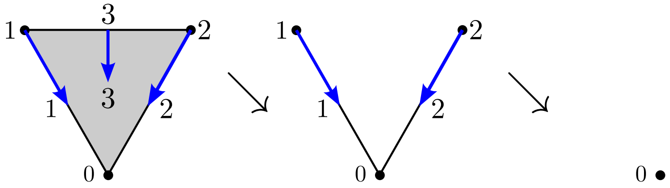
1. What about non-geodesic spaces? Finite metric spaces?
2. Connections to Ripser?

Discrete Morse theory (Forman 1998)



- Discrete Morse function $K \to \mathbb{R}$ with discrete gradient.

Discrete Morse theory (Forman 1998)

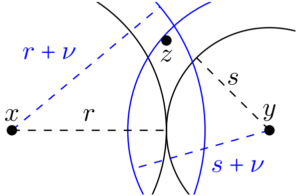


- Discrete Morse function $K \to \mathbb{R}$ with discrete gradient.
- They induce collapses that preserve the homotopy type.

Generalized contractibility lemma

The geodesic defect

Definition (Bonk, Schramm 2000). The metric space X is *ν-geodesic* if for all $x, y \in X$ and $r, s \geq 0$ with $r + s = d(x, y)$ there exists $z \in X$ with $d(x, z) \leq r + \nu$ and $d(y, z) \leq s + \nu$.



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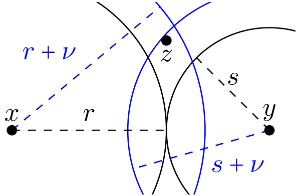

- $\nu \geq \frac{1}{2} \inf_{x \neq y} d(x, y)$
- X is r-geodesic if it is an r-dense subset of a geodesic metric space

Generalized contractibility lemma

Theorem (Bauer, R). Let X be a finite δ-hyperbolic metric space. Then there exists a discrete gradient encoding the collapses

$$\mathrm{Rips}_u(X) \searrow \mathrm{Rips}_t(X) \searrow \{*\}$$

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Sketch of proof

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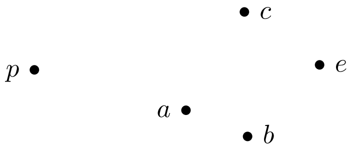

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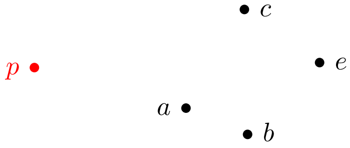

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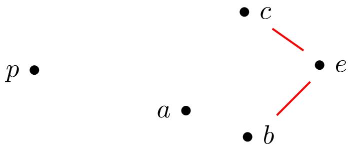

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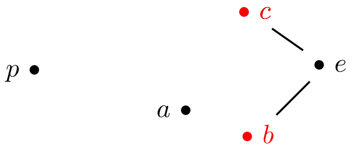

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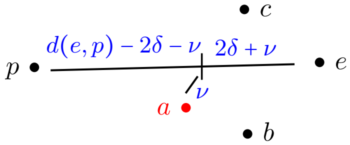
- Find $a \in X$: $d(a,p) \leq d(e,p) - 2\delta$ and $d(a,e) \leq 2\delta + 2\nu \leq t - 2\delta$



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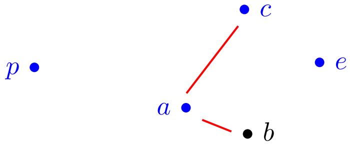

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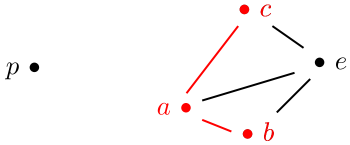
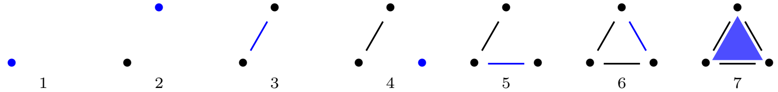
- The link of e is a cone with apex a and $\mathrm{Rips}_t(X) \searrow \mathrm{Rips}_t(X \setminus \{e\})$



Computing persistent homology



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R

$= D \cdot$

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V

Algorithm:

- while $\exists i < j$ with $\operatorname{pivot} R_i = \operatorname{pivot} R_j$
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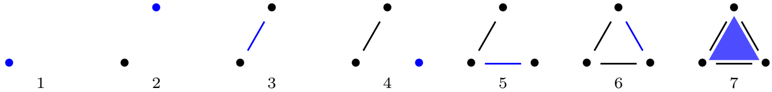

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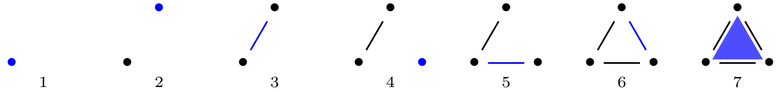

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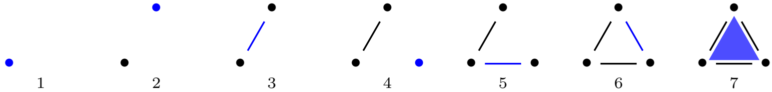

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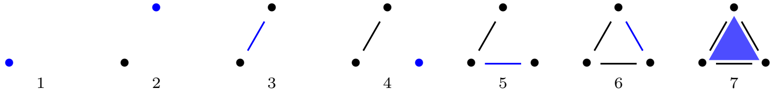

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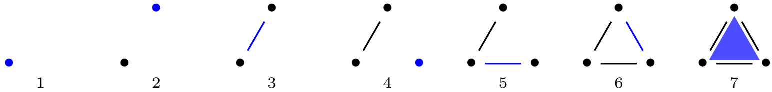

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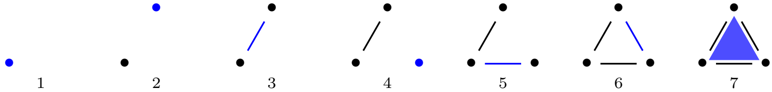

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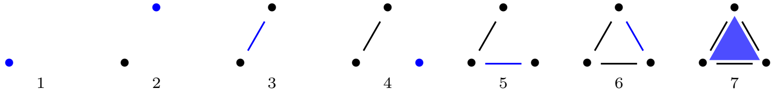

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Computing persistent homology

The diam-lexicographic filtration

We use the *lexicographic refinement* of the Vietoris–Rips filtration:

- choose a total order on the vertices
- order simplices by diameter
- order simplices with the same diameter lexicographically

Computing persistent homology

Apparent pairs

Ripser uses the following construction for a computational shortcut:

Definition. In a simplexwise filtration $(K_i = \{\sigma_1, \dots, \sigma_i\})_i$, a pair of simplices (σ, τ) is an *apparent pair* if

- σ latest proper face of τ, and
- τ is the earliest proper coface of σ.

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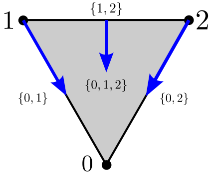
Lemma. If (σ_i, σ_j) is an apparent pair, then $[i, j)$ is an interval in the persistence barcode.

Apparent pairs

Lemma. The apparent pairs form a discrete gradient.



Apparent pairs

Discrete Morse theory

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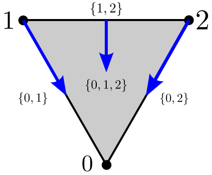

Remark. Kahle (2009) uses a specific apparent pairs gradient to study random Vietoris–Rips complexes above the thermodynamic limit.

Collapsing Rips complexes of trees

Let X be the path length metric space for a weighted tree $T = (V, E)$.

Collapsing Rips complexes of trees

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Theorem (Bauer, R). If X is ordered in a compatible way, the apparent pairs gradient induces a sequence of collapses

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for every $u > t > 0$ such that no edge $e \in E$ has length $l(e) \in (t, u]$.

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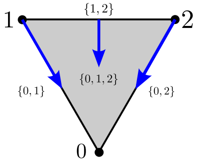
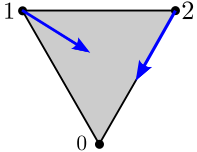
- Ripser computes the persistent homology of X without a single column operation.
- Explains Ripser's outstanding performance on genetic distances.

Generalized discrete Morse theory





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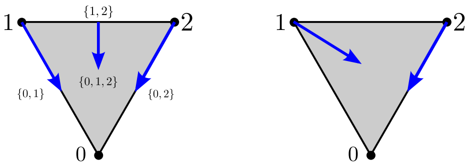

Definition. A monotone $f\colon K \to \mathbb{R}$ is a *generalized discrete Morse function* if there exists a partition of K, called the *discrete gradient* of f,

$$W = \{[\sigma, \tau] = \{\eta \mid \sigma \subseteq \eta \subseteq \tau\}\}$$

such that for $\sigma \subseteq \tau$ it is equivalent $f(\sigma) = f(\tau) \Leftrightarrow \sigma, \tau \in I \in W$.

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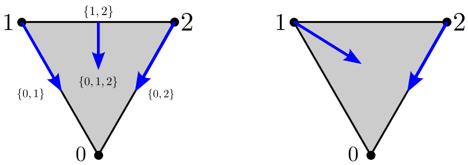

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Example (Bauer, Edelsbrunner 2016). For finite $X \subseteq \mathbb{R}^d$ in general spherical position the Čech radius function is a generalized discrete Morse function with its discrete gradient determined by smallest circumspheres.

Generalized discrete Morse theory

Refine W to another discrete gradient

$$\widetilde{W} = \{(\psi \setminus \{v\}, \psi \cup \{v\}) \mid \psi \in [\rho, \phi] \in W,\ v = \min(\phi \setminus \rho)\}$$

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Lemma. The zero persistence apparent pairs with respect to the f-lexicographic order are precisely the gradient pairs of $\widetilde{W}$.

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Some proof details

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- for *any* total order on V the apparent pairs gradient for the lexicographically refined Vietoris–Rips filtration also induces these

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- A detailed analysis shows that the apparent pairs gradient for the lexicographically refined Vietoris–Rips filtration induces these

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- Extended the Contractibility Lemma to finite metric spaces and made it filtration compatible.
- Identified a subclass of metric spaces for which the persistent homology computation is very efficient.