Gromov Hyperbolicity, Geodesic Defect, and Apparent Pairs in Vietoris–Rips Filtrations

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Abstract

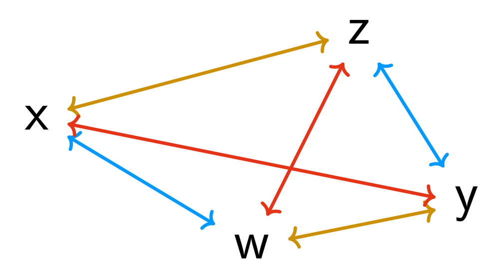
Motivated by computational aspects of persistent homology for Vietoris–Rips filtrations, we generalize a result of Eliyahu Rips on the contractibility of Vietoris–Rips complexes of geodesic spaces for a suitable parameter depending on the hyperbolicity of the space. We consider the notion of geodesic defect to extend this result to general metric spaces in a way that is also compatible with the filtration. We further show that for finite tree metrics the Vietoris–Rips complexes collapse to their corresponding subforests. We relate our result to modern computational methods by showing that these collapses are induced by the apparent pairs gradient, which is used as an algorithmic optimization in Ripser, explaining its particularly strong performance on tree-like metric data.

Background

Let X be a metric space and $t > 0$. The *Vietoris–Rips complex* is the abstract simplicial complex

$$\mathrm{Rips}_t(X) = \{\emptyset \neq S \subseteq X \mid S \text{ finite}, \ \mathrm{diam}\, S \leq t\}.$$

Definition. X is (Gromov) *δ-hyperbolic* if for all four points

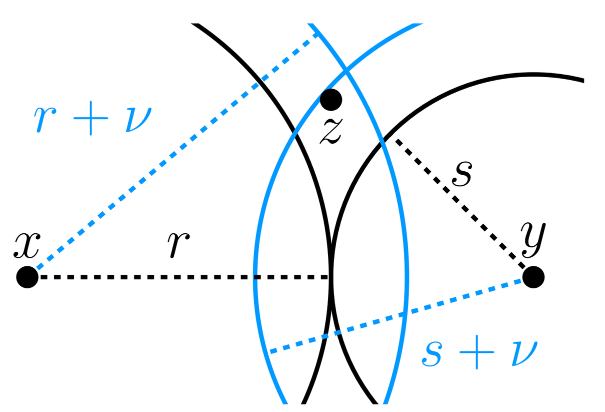


$$d(x,w) + d(y,z) \leq \max\{d(x,y) + d(z,w), d(x,z) + d(y,w)\} + 2\delta.$$

Lemma (Contractibility Lemma; Rips, Gromov [1]). Let X be a δ-hyperbolic geodesic metric space. Then the complex $\mathrm{Rips}_t(X)$ is contractible for every $t \geq 4\delta$.

The geodesic defect

Definition. The *geodesic defect* $\nu(X)$ of X is the infimum over all ν such that X is *ν-geodesic*, i.e., for all $x, y \in X$ and $r, s \geq 0$ with $r + s = d(x,y)$ there exists $z \in X$ such that



Properties.

- $\nu(X) \geq \frac{1}{2} \inf_{x \neq y} d(x,y)$
- Y a geodesic metric space and $X \subseteq Y$ an r-dense subset. Then, $\nu(X) \leq r$.

Generalized Contractibility Lemma

Let X be a finite δ-hyperbolic ν-geodesic metric space.

Theorem (Bauer, R)**.** There exists a filtration compatible discrete gradient that induces, for every $u > t \geq 4\delta + 2\nu$, the collapses

$$\mathrm{Rips}_u(X) \searrow \mathrm{Rips}_t(X) \searrow \{*\}.$$

Important special case. Tree metric space (V, d), where V is the vertex set of a positively weighted tree $T = (V, E)$. The geodesic defect is $\nu(V) = \frac{1}{2} \max_{e \in E} l(e)$ and V is 0-hyperbolic.

Apparent pairs

Choose a total order on X. A pair (σ, τ) of simplices in $\mathrm{Rips}_\infty(X)$ with the same diameter is a (zero persistence) *apparent pair* if

- σ is the lexicographically maximal facet of τ
- τ is the lexicographically minimal cofacet of σ.

Lemma. The collection of apparent pairs forms a discrete gradient.

Collapsing Rips complexes of trees

Let X be a tree metric space for a positively weighted tree $T = (V, E)$. Assume that X is sorted in a *compatible way*.

Theorem (Bauer, R)**.** The apparent pairs gradient induces, for every $u > t > 0$ such that no edge $e \in E$ has length $l(e) \in (t, u]$, the collapses

$$\mathrm{Rips}_u(X) \searrow \mathrm{Rips}_t(X) \searrow T_t,$$

where T_t is the graph on V with the edges in E of length at most t.

Experiments

In Ripser the reverse colexicographic order is used. If the input is a tree metric with the points ordered in reverse order of the distances to some arbitrarily chosen root, then

- Ripser will identify all non-tree simplices in apparent pairs
- Ripser computes its trivial persistent homology without a single column operation.

In practice, we observe that on data that is almost tree-like Ripser exhibits exceptionally good computational performance.

covid data (5000 points)	time for H_0 & H_1	non-apparent
compatible order	12s	4544
reversed order	6m 19s	19344
random order	2m 52s	13957

Our results provide a partial geometric explanation for this behavior and yield a heuristic for preprocessing tree-like data by sorting the points to speed up the computation in such cases.

Selected references

[1] Mikhael Gromov. Hyperbolic Groups. In S. M. Gersten, editor, Essays in Group Theory, Mathematical Sciences Research Institute Publications, pages 75–263. Springer. doi: 10.1007/978-1-4613-9586-7_3.

[2] Robin Forman. Morse theory for cell complexes. Adv. Math., 134(1):90–145, 1998. doi: 10.1006/aima.1997.1650.

[3] Ulrich Bauer and Fabian Roll. Gromov Hyperbolicity, Geodesic Defect, and Apparent Pairs in Vietoris–Rips Filtrations. Accepted to SoCG 2022. Extended version: arXiv:2112.06781.