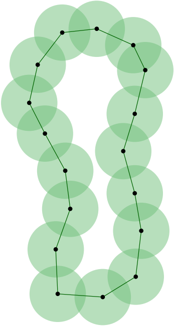
Bridging Persistent Homology and Discrete Morse Theory with Applications to Shape Reconstruction

Fabian Roll (TUM)

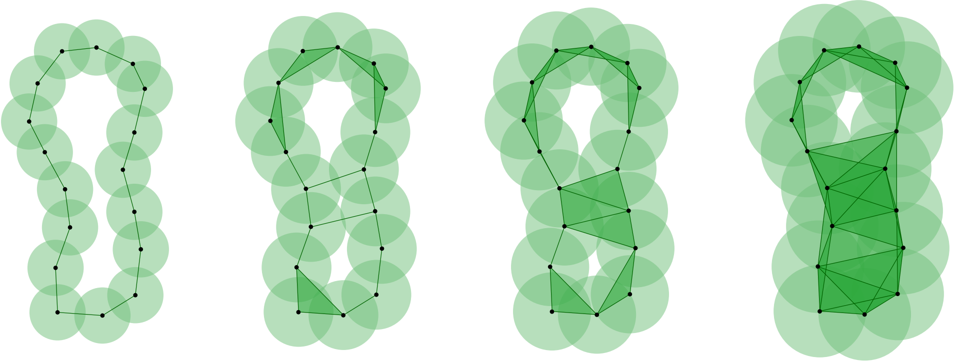
AATRN seminar

Joint work with Ulrich Bauer

Persistent homology



Persistent homology



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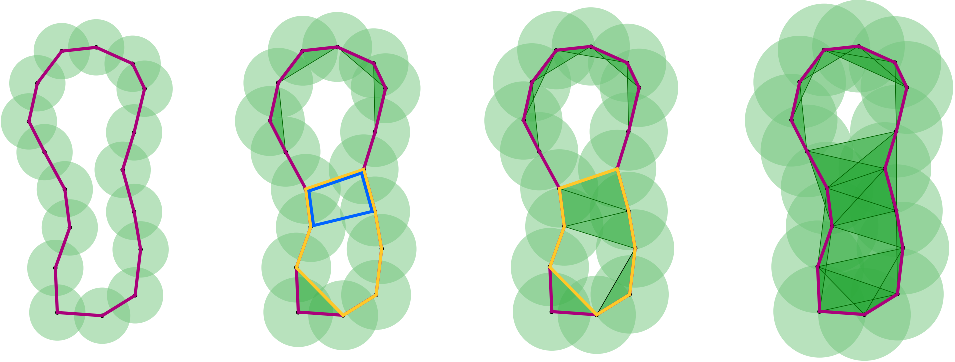

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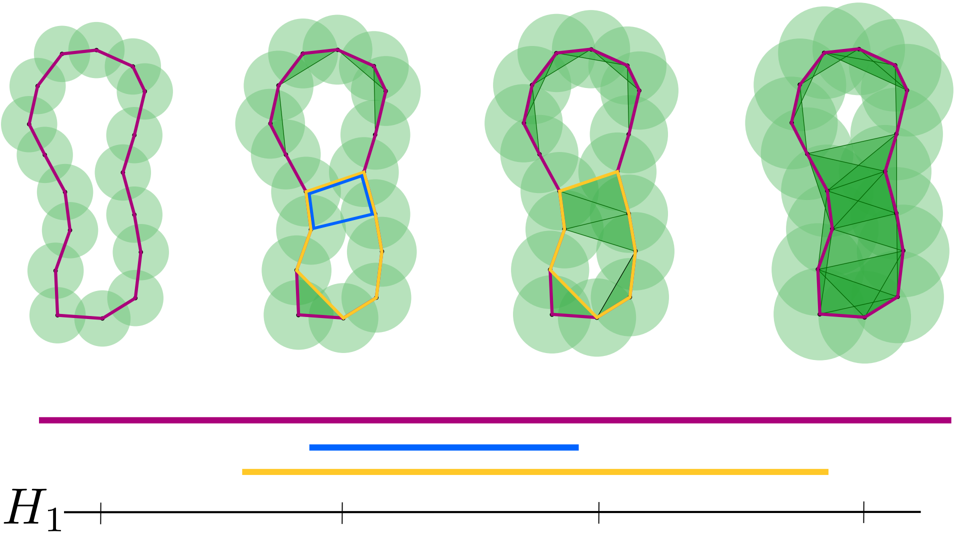
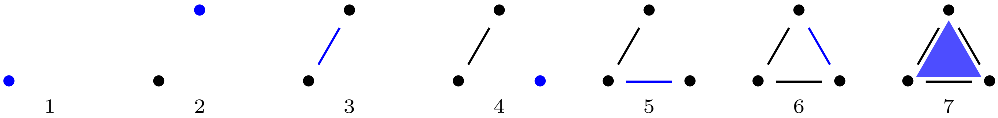


Computing persistent homology



R:

	1	2	3	4	5	6	7
1			1		1		
2			1			1	
3							1
4					1	1	
5							1
6							1
7							

$$R = D \cdot V$$

V:

	1	2	3	4	5	6	7
1	1						
2		1					
3			1				
4				1			
5					1		
6						1	
7							1

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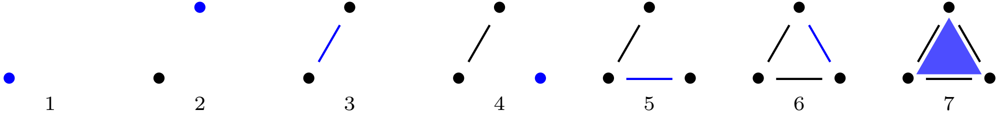

	1	2	3	4	5	6	7
1			1		1		
2			1			1	
3							1
4					1	1	
5							1
6							1
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R

$= D \cdot$

	1	2	3	4	5	6	7
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2		1					
3			1				
4				1			
5					1		
6						1	
7							1

V

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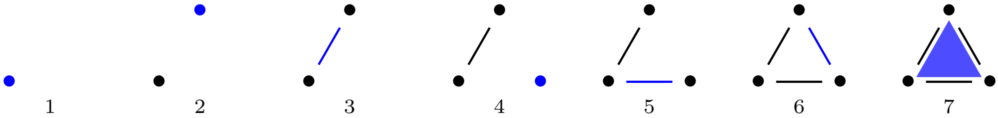

$R = D \cdot V$

R:

	1	2	3	4	5	6	7
1			1		1	1	
2			1			1	
3							1
4					1	0	
5							1
6							1
7							

V:

	1	2	3	4	5	6	7
1	1						
2		1					
3			1				
4				1			
5					1	1	
6						1	
7							1

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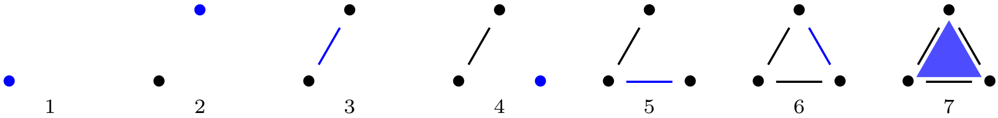

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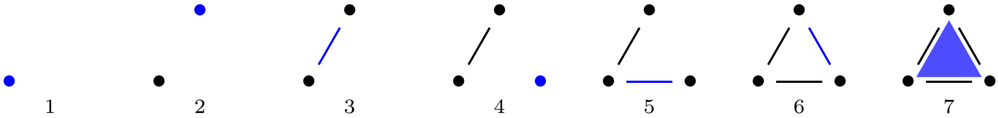

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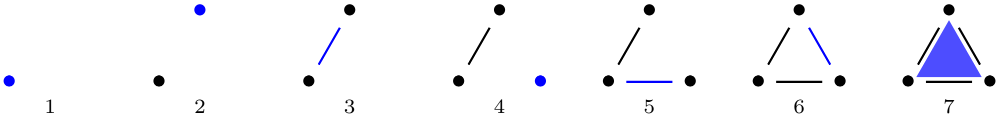

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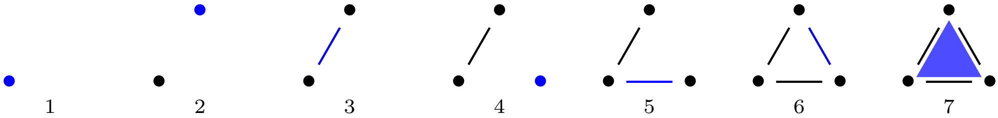

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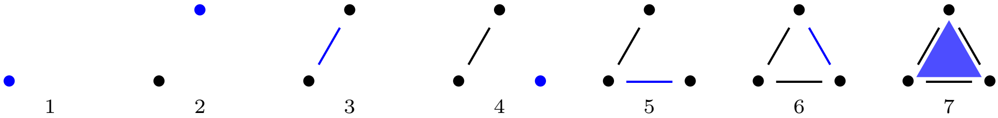

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V

We have the following construction for a computational shortcut:

Definition. In a simplexwise filtration $(K_i = \{\sigma_1, \dots, \sigma_i\})_i$, a pair of simplices (σ_i, σ_j) is an *apparent pair* if

- σ_i latest proper face of σ_j, and
- σ_j is the earliest proper coface of σ_i.

Computing persistent homology

Apparent pairs

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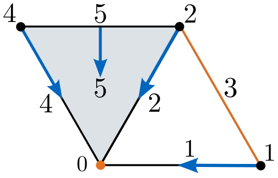
Lemma (Bauer). If (σ_i, σ_j) is an apparent pair, the interval $[i, j)$ is in the persistence barcode.

Discrete Morse theory

A *discrete Morse function* is a

- monotone function $f\colon K \to \mathbb{R}$ that
- partitions the complex into pairs and critical simplices, yielding the *discrete gradient V*



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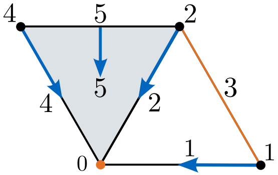
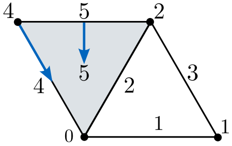


Discrete Morse functions - and their gradients - encode collapses:



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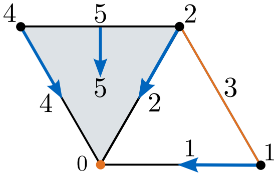

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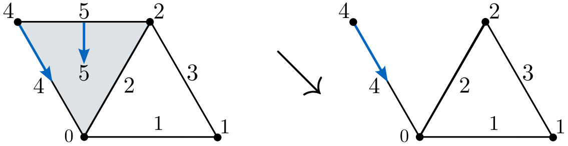

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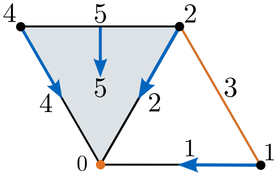

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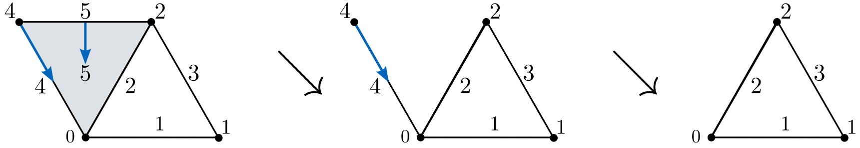
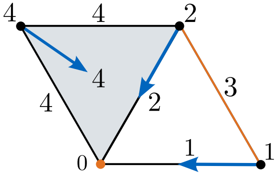


Generalized discrete Morse theory

Generalized gradients consist of intervals (in the face poset) instead of just facet pairs:



Persistent homology and discrete Morse theory

- Bauer/Lange/Wardetzky (2010):
  Optimal topological simplification of discrete functions on surfaces
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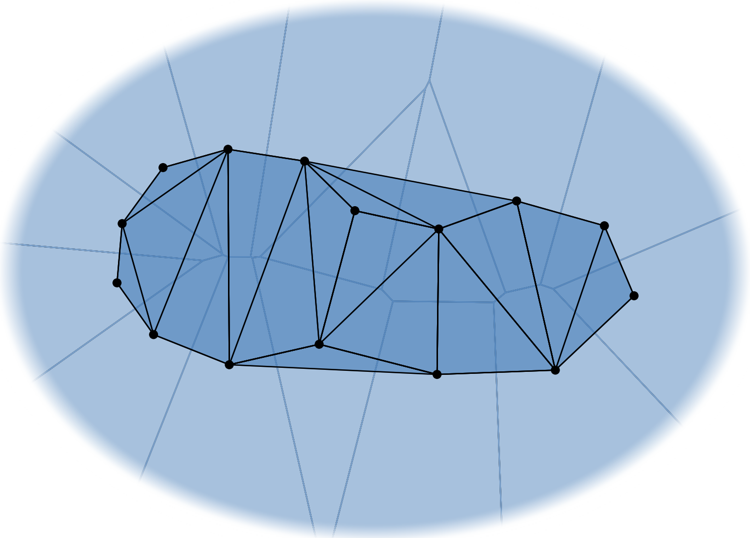
- Bauer/R (2022):
  Gromov hyperbolicity, geodesic defect, and apparent pairs in Vietoris–Rips filtrations
  - the zero persistence apparent pairs of a lexicographically refined sublevel set filtration of a generalized discrete Morse function refine the corresponding generalized gradient

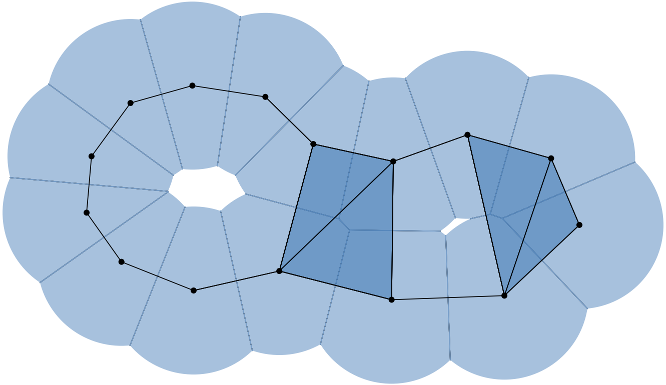
Delaunay complexes

Voronoi diagram and Delaunay triangulation



Delaunay complexes

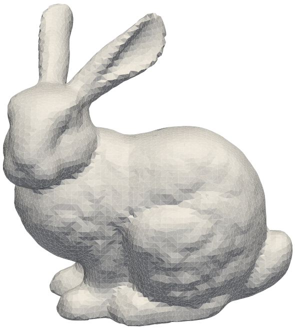
Definition. The *Delaunay complex* $\mathrm{Del}_r(X)$, or *alpha complex*, of $X \subseteq \mathbb{R}^d$ is the nerve of the cover by closed Voronoi balls of radius r centered at points in X.



Wrap

- Originally introduced by Edelsbrunner (1995) as a subcomplex of the Delaunay triangulation for surface reconstruction, using flow lines associated to Euclidean distance functions
- Redeveloped using discrete Morse theory (Forman 1998) by Bauer & Edelsbrunner (2014/17)

Delaunay complex



Wrap complex

Morse Theory of Čech and Delaunay complexes

Proposition (Bauer, Edelsbrunner 2014). The Čech and Delaunay radius functions are both generalized discrete Morse functions.

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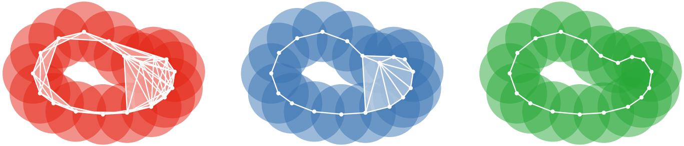
Theorem (Bauer, Edelsbrunner 2017).
Čech, Delaunay, and Wrap complexes (at any scale r) are related by collapses encoded by a single discrete gradient field:

$$\check{\mathrm{C}}\mathrm{ech}_r(X) \searrow \mathrm{Del}_r(X) \searrow \mathrm{Wrap}_r(X).$$



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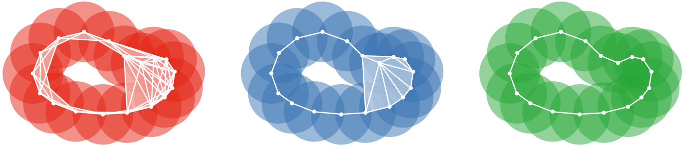
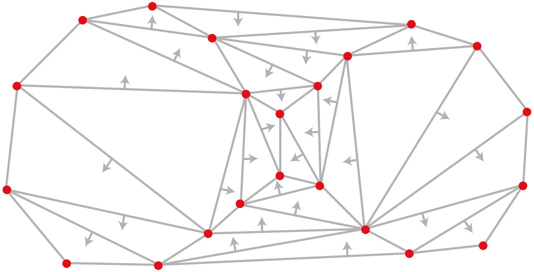


Remark. The Wrap complex $\mathrm{Wrap}_r(X)$ is the smallest subcomplex of $\mathrm{Del}_r(X)$ such that the Delaunay gradient induces a collapse $\mathrm{Del}_r(X) \searrow \mathrm{Wrap}_r(X)$.

Wrap complexes

Consider the gradient V of the Delaunay radius function $\mathrm{Del}(X) \to \mathbb{R}$.



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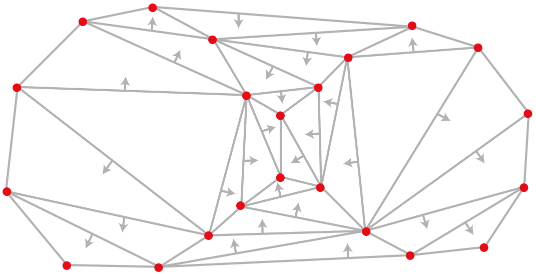

Definition (Edelsbrunner 1995; Bauer, Edelsbrunner 2017).
$\mathrm{Wrap}_r(X)$ is the *descending complex* of V on $\mathrm{Del}_r(X)$: smallest subcomplex of $\mathrm{Del}_r(X)$ that

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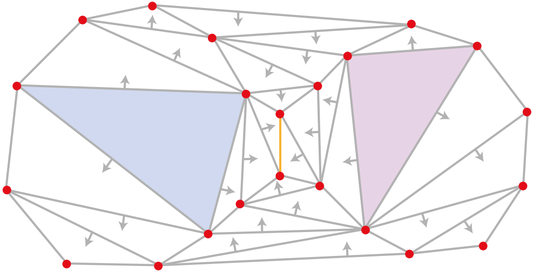

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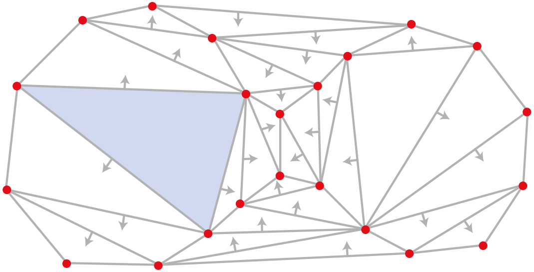

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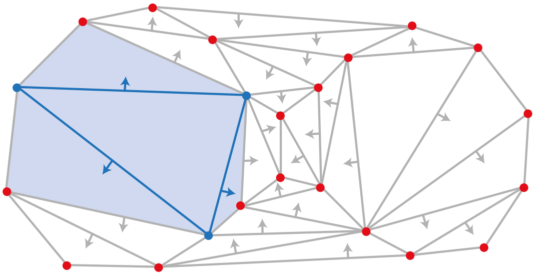

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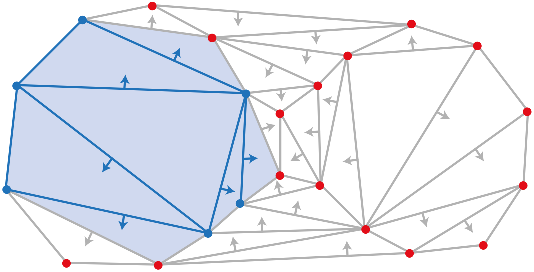

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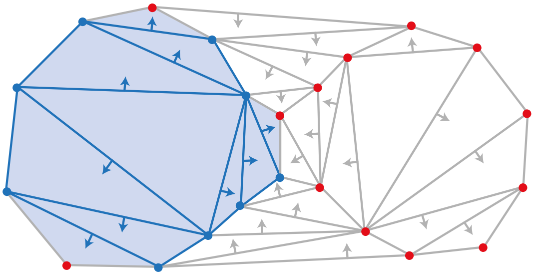

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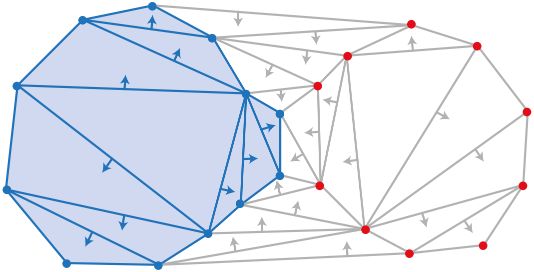

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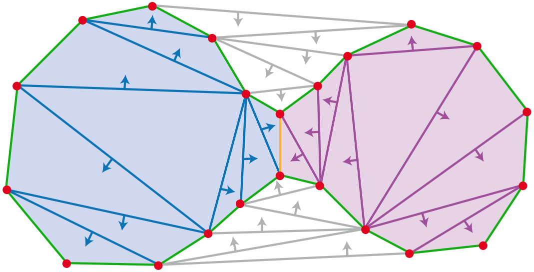

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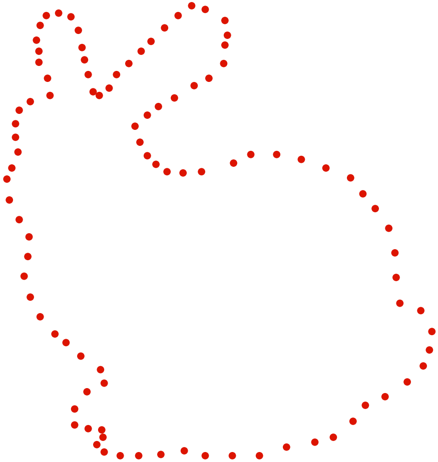
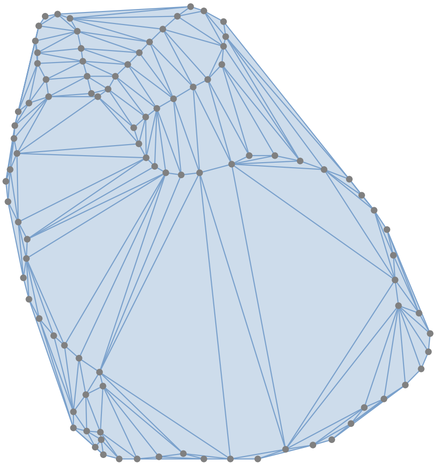


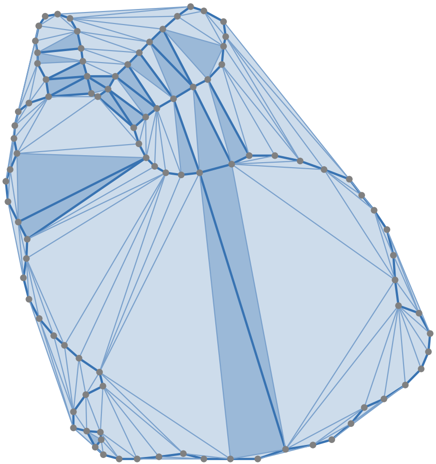
Point cloud

Wrap complexes



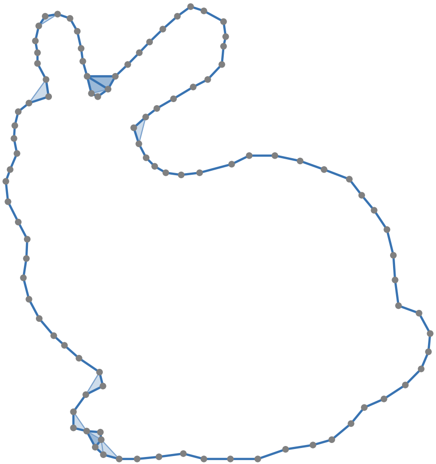
Delaunay triangulation

Wrap complexes



Critical simplices

Wrap complexes



Delaunay complex

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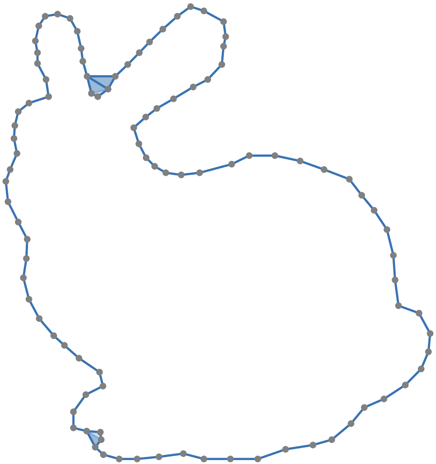

Wrap complex

Exhaustively reduced cycles

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- uses a variant of Gaussian elimination:
  - $R = D$ boundary matrix ($\mathbb{Z}_2$ coefficients) of $K_\bullet$,
    $V = I$ identity matrix

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- total order on X induces a lexicographic total order on the simplicies $\sigma_1 < \cdots < \sigma_n$ yielding a simplexwise refinement $K_\bullet = (K_i = \{\sigma_1, \ldots, \sigma_i\})_i$ of $\mathrm{Del}_\bullet(X)$
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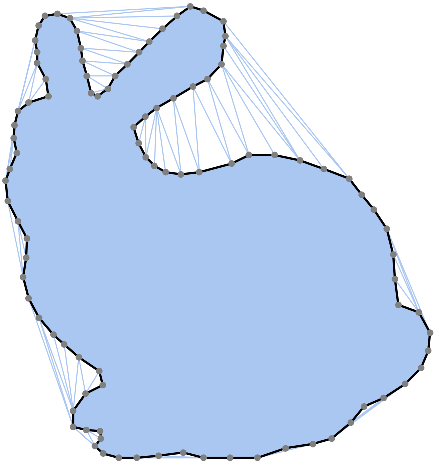

Reduction process

Algebraic gradient flows and persistent homology

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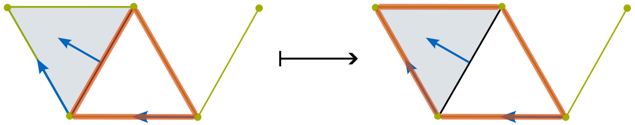

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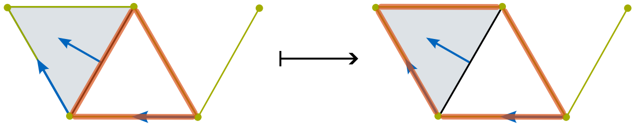

- Kozlov/Sköldberg/Jöllenbeck–Welker (2006/08/09) generalize discrete Morse theory to based chain complexes (*algebraic Morse theory*)
  - the basis elements take the role of the simplices in discrete Morse theory
  - all other notions translate straightforwardly

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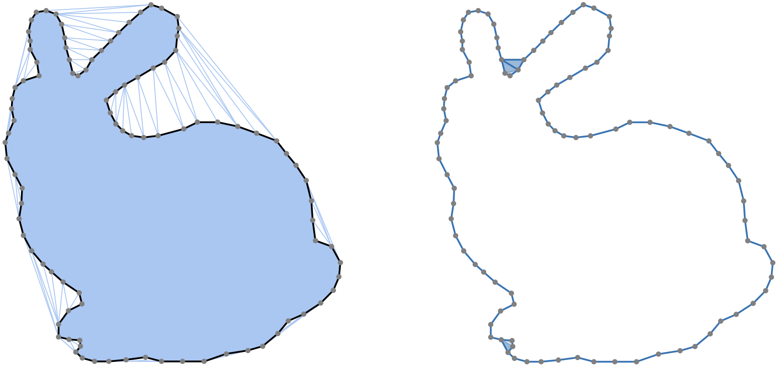
- persistence pairs form an algebraic gradient
- exhaustive Matrix reduction corresponds to gradient flow
- the lexicographically minimal cycles are invariant under the algebraic gradient flow
- connects to generalized discrete Morse theory, and hence to the Wrap complex, through gradient refinements (by apparent pairs)

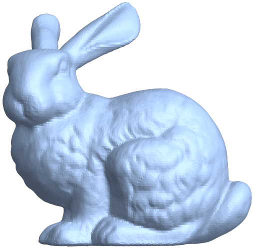
Minimal cycles and Wrap complexes

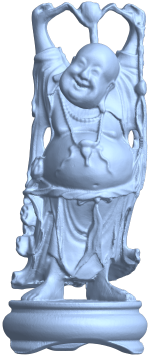
Theorem (Bauer, R). Let $X \subset \mathbb{R}^d$ be a finite subset in general position and let $r \in \mathbb{R}$. Then the lexicographically minimal cycles of $\mathrm{Del}_r(X)$, with respect to the Delaunay-lexicographic order on the simplices, are supported on $\mathrm{Wrap}_r(X)$.

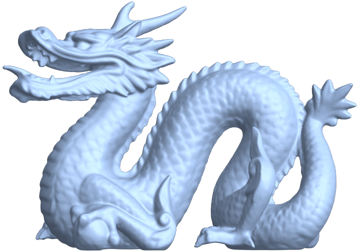


Point cloud reconstruction with most persistent features

The lexicographically minimal cycle, with respect to the Delaunay-lexicographic order on the simplices, corresponding to the interval in the persistence barcode of the Delaunay filtration with the largest death/birth ratio:







```
$ docker build -o output github.com/fabian-roll/wrappingcycles
```

Persistence pairs form an algebraic gradient

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Remark. For a simplexwise refined sublevelset filtration of a discrete Morse function, the corresponding discrete gradient is *extended by* the reduction gradient.

Exhaustive Matrix reduction corresponds to gradient flow

Definition. The *flow* $\Phi\colon C_* \to C_*$ determined by W is the chain map given by

$$\Phi(c) = c + \partial\,\mathrm{F}(c) + \mathrm{F}(\partial c),$$

where $\mathrm{F}\colon C_* \to C_{*+1}$ is the unique linear map defined on the basis elements $\sigma \in \Sigma_*$ as

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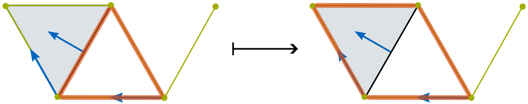
- If c is a cycle, then the flow reduces to $\Phi(c) = c + \partial\,\mathrm{F}(c)$ and therefore acts on each homology class of the chain complex by a change of representative cycle



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- the stablized cycle $\Phi^\infty(c)$ is the corresponding exhaustively reduced cycle

Bridging Persistent Homology and Discrete Morse Theory

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  - such that the corresponding algebraic gradient flow can be viewed as a variant of the reduction algorithm for computing persistent homology
- Establish a strong connection between Morse-theoretic and homological approaches to shape reconstruction
  - lexicographically minimal cycles of $\mathrm{Del}_r(X)$ are supported on the Wrap complex $\mathrm{Wrap}_r(X)$

Funding: DFG – TRR109 *Discretization in Geometry and Dynamics*